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ON THE INITIAL STAGES OF VORTEX WAVE INTERACTIONS IN HIGHLY CURVED BOUNDARY LAYER FLOWS

Philip Hall[1]
Department of Mathematics
University of Manchester
Manchester M13 9PL
UNITED KINGDOM

ABSTRACT

The nonlinear interaction equations describing vortex-Rayleigh wave interactions in highly curved boundary layers are derived. These equations describe a strongly nonlinear interaction between an inviscid wave system and a streamwise vortex. The coupling between the two structures is quite different than that found by Hall and Smith (1991) in the absence of wall curvature. Here the vortex is forced over a finite region of the flow rather than in the critical layer associated with the wave system. When the interaction takes place the wave system remains locally neutral as it moves downstream and it's self interaction drives a vortex field of the same magnitude as that driven by the wall curvature. This modification of the mean state then alters the wave properties and forces the wave amplitude to adjust itself in order that the wave frequency is constant. Solutions of the interaction equations are found for the initial stages of the interaction in the case when the wave amplitude is initially small. Our analysis suggests that finite amplitude disturbances can only exist when the vortex field is finite at the initial position where the interaction is stimulated.

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1. Introduction

Our concern is with the interaction of Rayleigh waves and streamwise vortices in pressure gradient driven boundary layers on highly curved walls. The first step is to derive the coupled strongly nonlinear interaction equations governing the small wavelength vortex field driven by small amplitude Rayleigh waves. The interaction equations we obtain differ from those found previously by Hall and Smith (1991). In the latter paper it was found that three-dimensional Rayleigh waves drive the vortex field in the critical layer. The vortex field then acts back on the wave indirectly because the wave satisfies a modified form of Rayleighs equation which depends on the spanwise variation of the mean flow. Here we find that the Rayleigh waves cannot drive the vortex field in the critical layer. Instead we find that the forcing is distributed over the region where the vortex activity occurs, this enables us to reduce the interaction problem to a nonlinear partial differential system independent of the spanwise variable. We construct the small amplitude solutions of this system in two situations and show that the possible emergence of the vortex-wave interaction is sensitively controlled by the nature of the incoming mean profile.

The different stages of vortex-wave interactions associated with viscous (Tollmien-Schlichting) travelling waves are described in the papers by Hall and Smith (1988, 1989, 1990, 1991), Smith and Walton (1989), Bassom and Hall (1989), Bennett, Hall and Smith (1991) and Blackaby (1991). Typically it is found that Tollmien-Schlichting waves force the vortex field in a thin viscous layer at the wall for external flows and away from the walls for fully developed internal flows. For external flows the forcing results in an inhomogeneous boundary condition for the vortex velocity component in the spanwise direction. The situation with Rayleigh waves is more complicated and Hall and Smith (1991) showed that Rayleigh waves force the vortex in the critical layer associated with the wave. More precisely the wave induces a finite jump in the gradient of the spanwise velocity component of the vortex across the critical layer. The solution of the interaction equations in this case is made difficult by the spanwise and streamwise variation of the critical layer position; as yet no numerical solutions of the interaction equations derived by Hall and Smith have been obtained.

In the present paper we examine the Rayleigh wave-vortex interaction problem in the case of highly curved boundary layers. This regime corresponds to high values of the Görtler number associated with the flow and we assume that the vortex wavelength is small compared to the boundary layer thickness. The latter assumption enables us to make use of the small wavelength large amplitude structure of vortices discussed by Hall and Lakin (1988). The latter calculation was a development of the linear and weakly nonlinear investigations of the Görtler vortex mechanism by Hall (1982a,b). In the present calculation we show that at small wavelengths the critical layer forcing of the vortex by Rayleigh waves becomes exponentially small compared to a new distributed mechanism operational over a finite part of the boundary layer. We shall see that the forcing is confined to the finite part of the boundary layer where vortex activity occurs.

The approach of Hall and Lakin (1988) applied to the present problem shows that the interaction equations can be significantly reduced to a stage where the spanwise dependence of the disturbances is described analytically. The means that the strongly nonlinear interaction equations for vortex-wave interactions at small wavelengths can be reduced to a partial differential system dependent only on the two-dimensional boundary layer variables. These interaction equations are derived in §2.

A non-trivial problem associated with the equations governing vortex-wave interactions is the determination of appropriate initial conditions for the equations. In §3 we tackle the problem for the case when a Rayleigh wave of small amplitude is generated from a finite amplitude vortex field. We shall derive constraints on the vortex field which allow for the emergence of a vortex-wave interaction. In §4 we investigate the possibility of the spontaneous generation of Rayleigh waves and vortices at some position in the streamwise direction. Our results suggest that the stringent constraints which must be satisfied if such an interaction is to take place virtually rule out the spontaneous generation of Rayleigh waves and vortices. In §5 we discuss the results of §3,4 and draw some conclusions.

2. Derivation of the Vortex-Wave Interaction Equations for Highly Curved Flows

We consider the flow of a viscous incompressible fluid of density ρ, kinematic viscosity ν past a rigid wall defined by $y = 0$ with respect to a coordinate system (x, y, z) with x, y, z measuring distance along the wall, normal to the wall and in the spanwise direction respectively. We assume that x, y, z have been made dimensionless using the lengths $L, LR^{-1/2}, LR^{-1/2}$ where L is a lengthscale in the x-direction and R is defined by

$$R = \frac{U_0 L}{\nu}. \tag{2.1}$$

Here U_0 is a typical fluid speed in the x direction and the Reynolds number R is taken to be large. If we take the wall curvature to be $a^{-1}\chi(x)$ then the equations to determine the combined Görtler vortex-mean flow are

$$\begin{gathered}\boldsymbol{\nabla} \cdot \boldsymbol{u}^* = 0, \\ \boldsymbol{u}^* \cdot \nabla \boldsymbol{u}^* + \begin{pmatrix} 0 \\ \chi G u^{*2}/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\overline{p}_x \\ -p_y^* \\ -p_z^* \end{pmatrix} + \Delta \boldsymbol{u}^*. \end{gathered} \tag{2.2}$$

In the above equations G is the Görtler number defined by

$$G = R^{1/2}\frac{a}{L},$$

which is held fixed in the limit $R \to \infty$, $\frac{\rho U_0^2 \overline{p}_x}{L}$ is the streamwise pressure gradient at the edge of the boundary layer and u^*, v^*, w^*, p^* have been made dimensionless using the scales $U_0, U_0 R^{-1/2}, U_0 R^{-1/2}$ and $\rho U_0^2 R^{-1}$ respectively. Finally the operator Δ appearing in (2.2) is the two-dimensional Laplacian $\partial_y^2 + \partial_z^2$. Thus streamwise diffusion is negligible in (2.2) since it operates on a longer lengthscale than diffusion in the y, z directions. In addition, p_x^* is negligible so that (2.2) is parabolic in x, therefore no mechanism to produce upstream influence is present.

In order to study the influence of vortex flows on the inviscid instability of shear flows Hall and Horseman (1991) superimposed on the flow an infinitesimally small Rayleigh wave.

Following these authors we therefore replace the functions u^*, v^*, w^* used in the derivation of (2.2) by writing

$$\begin{aligned} u^* &= u(x,y,z) + \tfrac{\delta}{R^{1/2}}\left[U(x,y,z)E + \text{ complex conjugate}\right] + \cdots, \\ v^* &= v(x,y,z) + \delta\left[V(x,y,z)E + \text{ complex conjugate}\right] + \cdots, \\ w^* &= w(x,y,z) + \delta\left[W(x,y,z)E + \text{ complex conjugate}\right] + \cdots, \end{aligned} \tag{2.3}$$

where

$$E = \exp iR^{1/2}\left\{\int^x \alpha(x)dx - \Omega t\right\}. \tag{2.4}$$

Thus we have assumed that the disturbance, of arbitrarily small size δ, is periodic in time (with t scaled on $R^{-1/2}U_0^{-1}$) and varies on a short $O(R^{-1/2})$ lengthscale in the x direction. Note that if δ is sufficiently small then u, v, w satisfy (2.2) with the asterisks removed. The equations satisfied by the disturbance in the limit $R \to \infty$ are found to be

$$\begin{aligned} &i\alpha U + V_y + W_z = 0, \\ &i\alpha(u-c)U + Vu_y + Wu_z = -i\alpha P, \\ &i\alpha(u-c)V = -P_y, \\ &i\alpha(u-c)W = -P_z, \end{aligned} \tag{2.5}$$

with $c = c(x) = \Omega/\alpha$ and P denoting the pressure perturbation corresponding to U, V, W. A more convenient form of (2.5) is obtained by eliminating the velocity field U, V, W to give the pressure form of the Rayleigh equation for longitudinal vortex flows; we thus obtain

$$\left(\frac{P_y}{(u-c)^2}\right)_y + \left(\frac{P_z}{(u-c)^2}\right)_z - \frac{\alpha^2 P}{(u-c)^2} = 0, \tag{2.6a}$$

and this must be solved with P periodic in z and subject to

$$\begin{aligned} P_y = 0, \quad y = 0, \\ P \to 0, \quad y \to \infty. \end{aligned} \tag{2.6b, c}$$

For a given velocity field $u(x,y,z)$ the eigenvalue problem (2.6) can be solved for $\alpha = \alpha(\Omega, c)$, and the flow is unstable if eigenvalues can be found with $\alpha_i < 0$. A similar eigenvalue problem for the temporal instability problem with α real and c complex can also be used to classify a given flow as stable or unstable. Using the latter approach Hall and Horseman (1991) were able to show that Görtler vortices cause a Blasius boundary layer to become inviscidly unstable when they become of sufficiently large amplitude. The results found by the latter authors were in excellent agreement with the experimental observations of Swearingen and Blackwelder (1987).

In the context of vortex-wave interaction theory interest centres on the neutral solutions of (2.6) which have $\alpha = \alpha(x)$, $\alpha c = \Omega =$ constant and $\delta = \delta(x)$ chosen to be large enough to maintain the wave in this neutral state as it moves downstream. In Hall and Smith (1991), hereafter referred to as HS, it was shown that the forcing takes place in the critical layer and leads to a jump in w_y across the layer. However the expression for the jump in w_y vanishes if P is independent of z at the critical layer. This would for example be the case in the degenerate case $u = u(x, y)$ which applies when no vortices are present in the flow. Thus a fundamental property of the interaction is that it can only occur when the streamwise flow has a spanwise dependence.

In fact there exists a class of flows which support large amplitude streamwise vortices over only a finite part of a boundary layer. If the critical layer associated with these flows is in a region where no vortex activity occurs then the mechanism described in HS cannot operate. The class of flows referred to above corresponds to the strongly nonlinear Görtler vortex flows first discussed by Hall and Lakin (1988). These flows correspond to the limit $\frac{\partial}{\partial z} \gg 1$ (or equivalently $x \gg 1$) and have the structure sketched in Figure (1.1).

In Regions I and II there is no vortex activity and at zeroth order the mean state simply satisfies the equations governing the unperturbed boundary layer. In Region III, which corresponds to $y_1 < y < y_2$, a finite amplitude vortex exists and drives the mean state. The vortex activity is reduced exponentially to zero in thin shear layers surrounding y_1 and y_2. It follows that if we have a situation in which the critical layer of the system (2.6) associated with the above flow is below $y = y_1$, then an alternative mechanism to that proposed by HS must be found. For the moment we assume that $u(x, y, z)$ appearing in (2.2), with (u^*, v^*, w^*, p^*) replaced by (u, v, w, p) has a region adjacent to $y = 0$ where it is independent of z and that the critical layer occurs in this region. We now have a situation where the Rayleigh wave satisfying (2.6) cannot force the vortex flow in the manner discussed in HS in the region of vortex activity.

In fact the vortex-wave interaction in such a case occurs in a simpler manner than that found in HS. In order to allow the Rayleigh wave to drive the vortex we let δ in (2.3) increase until the nonlinear terms involving U, V, W in the y and z momentum equations become comparable with those present for the vortex field. It is easy to see that this occurs when $\delta = R^{-1/2}\overline{\delta}$ with $\overline{\delta}$ of order R^0. In this case the momentum equations in (2.2) are modified to give

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \begin{pmatrix} 0 \\ \chi G u^2/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\overline{p}_x \\ -p_y \\ -p_z \end{pmatrix} + \Delta u - \overline{\delta}^2 \begin{pmatrix} 0 \\ -i\alpha U\overline{V} + V\overline{V}_y + W\overline{V}_z + \text{ c.c.} \\ -i\alpha U\overline{W} + V\overline{W}_y + W\overline{W}_z + \text{ c.c.} \end{pmatrix}, \quad (2.7)$$

where 'c.c.' denotes 'complex conjugate.'

Thus the vortex field is now driven by the Rayleigh wave through the forcing terms on the right hand side of (2.7). The Rayleigh wave is determined by (2.6) and so is itself a function of $u(x, y, z)$, thus there is a strongly nonlinear coupling between the vortex and the wave fields. In order that the Rayleigh wave remains neutral as it moves downstream the wavespeed adjust itself to the fluid speed at the inflection point. If the frequency is to remain fixed then there is an apparent difficulty because the wavespeed and wavenumber are effectively fixed by the previous consideration. The required extra degree of freedom is found by allowing $\overline{\delta}$ to adjust itself so as to modify the mean flow in such a way that the frequency remains fixed.

We note that (2.7) fails if $u = c$ in the region of vortex activity and in this case we return to the situation discussed by HS. A further important point to notice here is that the size of the Rayleigh wave, $O(R^{-1/2})$, required to drive the interaction is so small that the Rayleigh wave critical layer, of depth $O(R^{-1/6})$, remains linear.

We now indicate how the short wavelength asymptotic structure given by Hall and Lakin (1988) can be modified to take account of the forcing terms arising from the interaction of the Rayleigh wave with itself. We assume that the vortex activity is confined to the region $y_1 < y < y_2$. Here we take the vortex wavenumber to be large and expand u, v, w, and p appearing in (2.7) in the form

$$\begin{aligned} u &= \overline{u}_0 + \tfrac{1}{k}\{u_1 + U_1 C_1\} + O(-1/k^2), \\ v &= kV_1C_1 + \overline{v}_0 + V_2C_2 + O(1/k), \\ w &= W_1S_1 + O(1/k), \\ p &= k^4p_0 + k^3p_1 + k^2p_2 + k\{p_1 + C_1P_1\} + \cdots . \end{aligned} \qquad (2.8a, b, c, d)$$

Here $C_n = \cos nkz$, $S_n = \sin nkz$ and the functions $\overline{u}_0, U_0$, etc. depend only on x and y. The Rayleigh wave velocity components U, V, W appearing in (2.7) may be found from the large k solution of (2.5). In order to determine this structure we expand the wavespeed c in the form

$$c = c_0 + c_1/k + \cdots \qquad (2.9)$$

and a similar expansion is written down for α. We assume that $\bar{u}_0 \neq c_0$ in $y_1 < y < y_2$. We then write

$$G = \tilde{G}k^4 + \cdots, \overline{\delta} = k^2.$$

The appropriate expansions for U, V, W and P are

$$\begin{aligned} U &= \tilde{U}_0 + \tfrac{1}{k}\tilde{U}_1C_1 + \cdots, \\ V &= \tilde{V}_0 + \tfrac{1}{k}\tilde{V}_1C_1 + \cdots \\ W &= \tfrac{1}{k^2}\tilde{W}_1S_1 + \cdots \\ P &= \tilde{P}_0 + \cdots + \tfrac{1}{k^3}\tilde{P}_1C_1 + \cdots . \end{aligned} \qquad (2.10a, b, c, d)$$

If we substitute the above expansions into (2.5) we find that $\tilde{P}_0$ satisfies the usual Rayleigh pressure equation

$$\tilde{P}_{0yy} - \frac{2\overline{u}_{0y}}{\overline{u}_0 - c_0}\tilde{P}_{0y} - {\alpha_0}^2\tilde{P}_0 = 0, \qquad (2.11a)$$

where $\tilde{V}_0$ is given by

$$i\alpha_0(\overline{u}_0 - c_0)\tilde{V}_0 = -\tilde{P}_{0y}. \qquad (2.11b)$$

At next order we find that

$$i\alpha_0\tilde{U}_1 = -2\left\{\frac{\tilde{V}_0 U_{1y}}{\overline{u}_0 - c_0} - \frac{\overline{u}_{0y} U_1 \tilde{V}_0}{(\overline{u}_0 - c_0)^2}\right\} - \tilde{V}_{1y},$$

$$\tilde{V}_1 = -\frac{\tilde{V}_0 U_1}{\overline{u}_0 - c_0},$$

$$\tilde{W}_1 = \frac{2}{\overline{u}_0 - c_0}\left\{\tilde{V}_0 U_{1y} - \frac{\overline{u}_{0y} U_1 \tilde{V}_0}{\overline{u}_0 - c_0}\right\},$$

$$\tilde{P}_1 = 2i\alpha_1\left\{\tilde{V}_0 U_{1y} - \frac{\overline{u}_{0y} U_1 \tilde{V}_0}{\overline{u}_0 - c_0}\right\}.$$

We now substitute (2.8) into (2.7) and equate the leading order terms proportional to $C_1 = \cos kz$ in the x and y momentum equations. This yields

$$V_1 \overline{u}_{0y} = -U_1$$

$$\chi \tilde{G} \overline{u}_0 U_1 - \frac{4|\tilde{V}_0^2|_y U_1}{\overline{u}_0 - c_0} = -V_1.$$

These equations have a consistent solution only if $\overline{u}_0, \tilde{V}_0$ satisfy

$$\chi \tilde{G} \overline{u}_0 \overline{u}_{0y} - \frac{4|\tilde{V}_0^2|_y \overline{u}_{0y}}{\overline{u}_0 - \overline{c}_0} = 1. \tag{2.12}$$

This equation has to be solved in conjunction with the Rayleigh equation (2.11a). In fact it is convenient to write (2.11b) in terms of $\tilde{P}_0$ using (2.16) to give

$$\chi \tilde{G}(\overline{u}_0^2)_y/2 - \frac{4\overline{u}_{0y}}{\alpha_0^2(\overline{u}_0 - c_0)}|\frac{\tilde{P}_{0y}^2}{(\overline{u} - c_0)^2}|_y = 1. \tag{2.13}$$

In fact, since (2.11a) does not involve derivatives with respect to x, we can multiply $\tilde{P}_0$ by an arbitrary function of x, we therefore write $\tilde{P}_0 = \alpha_0 \frac{B(x)}{2} \tilde{\tilde{P}}_0$ where the eigenfunction $\tilde{\tilde{P}}_0$ satisfies (2.11a) together with

$$\tilde{\tilde{P}}_{0y} = 0, \ y = 0, \tilde{\tilde{P}}_0 \to 0, \ y \to \infty, \tag{2.14}$$

and some normalization condition. We note also that without any loss of generality we can take B, P_0 to be real. We then write (2.13) as

$$\chi \tilde{G}(\overline{u}_0^2)_y/2 - \frac{\overline{u}_{0y} B^2}{\overline{u}_0 - c_0}\left\{\frac{\tilde{\tilde{P}}_{0y}^2}{(\overline{u}_0 - c_0)^2}\right\}_y = 1. \tag{2.15}$$

Thus the outcome of the Rayleigh wave forcing is to introduce the term proportional to B^2 in the local mean flow equation (2.15). This in effect means that $\overline{u}_0$ and $\tilde{\tilde{P}}_0$ can only be obtained numerically because (2.15) is coupled to (2.11). As in Hall and Lakin (1988) we note that the equations obtained by equating leading order vortex terms proportional to C_1 do not determine the vortex. The required equation is obtained by equating leading order terms independent of z in the x momentum equations, this gives

$$\overline{u}_0\overline{u}_{0x} + \overline{v}_0\overline{u}_{0y} - \overline{u}_{0yy} + \overline{p}_x = \frac{\partial}{\partial y}\left(\overline{u}_0 V_1^2\right).$$

This equation can be integrated to give V_1^2 in the core, if the vortex is to vanish at y_1, y_2 we must have

$$\int_{y_1}^{y_2} (\overline{u}_0\overline{u}_{0x} + \overline{v}_0\overline{u}_{0y} - \overline{u}_{0yy} + \overline{p}_x)dy = 0.$$

We shall now summarize the interaction equations and matching conditions which the mean state and Rayleigh wave must satisfy if a vortex-wave interaction is to take place.

For convenience we now denote the zeroth order mean flow in Regions I, II, and III by $\boldsymbol{u}(x,y) = (u(x,y), v(x,y), 0)$. We also denote the Rayleigh wavenumber and wavespeed at zeroth order by α, c respectively. The zeroth order problem for the mean flow driven by the vortex-wave interaction can then be written down in the form:

Regions I, II

$$\left.\begin{aligned} u_x + v_y &= 0, \\ uu_x + vu_y &= -\overline{p}'_x + u_{yy}, \end{aligned}\right\} \tag{2.16a, b}$$

Region III

$$\left.\begin{aligned} &u_x + v_y = 0, \\ &u_y\left\{\chi G u - \tfrac{B^2}{u-c}\left\{\tfrac{P'^2}{(u-c)^2}\right\}'\right\} = 1, \end{aligned}\right\} \tag{2.17a, b}$$

$$\int_{y_1}^{y_2} \{uu_x + vu_y - u_{yy} + \overline{p}_x\}\, dy = 0, \tag{2.18}$$

together with the boundary conditions

$$\left.\begin{aligned} u = v = 0, &\quad y = 0, \\ u \to u_e, &\quad y \to \infty, \end{aligned}\right\} \tag{2.19}$$

and matching conditions

$$u, u_y, v, P, P_y \text{ continuous at } y = y_1, y_2. \tag{2.20}$$

We need not write down separate equations satisfied by P in I, II, III since, using the notation used above, the single equation

$$(u-c)(P_{yy} - \alpha^2 P) - 2u_y P_y = 0, \tag{2.21}$$

together with

$$P = 1,\ y = y_c,\ P_y = 0,\ y = 0,\ P = 0,\ y = \infty, \tag{2.22}$$

where y_c is the critical layer location are sufficient to determine P. Finally we note that $\alpha(x), c(x)$ the eigenvalues of (2.21), (2.22) must satisfy

$$\alpha c = \text{ constant},$$

so that the wave frequency stays fixed as it moves downstream. In effect this constraint is satisfied by adjusting the wave amplitude $B(x)$ appropriately as x varies. We further note that thin shear layers of thickness $k^{\frac{-2}{3}}$ are needed near y_1, y_2 in order to allow the vortices to decay to zero exponentially. These layers are passive and require only minor modifications to the structure given by Hall and Lakin (1988). An analysis in these shear layers shows that below y_1 and above y_2 the vortices are smaller by a factor e^{-k} raised to some positive power. This means that the criical layer forcing mechanism discussed by HS is exponentially small compared to the distributed mechanism discussed here.

Before investigating the growth of small amplitude Rayleigh waves from some position along the flow direction we first discuss briefly how the eigenrelation associated with (2.21), (2.22) can be calculated when u changes by a small amount. We shall assume then that u can be written as

$$u = \overline{u}(y) + \tilde{u}(y),$$

and we suppose $\bar{u}''$ is discontinuous at y_d. This discontinuity is associated with the transition layers at y_1, y_2. In these layers the jump in $\bar{u}''$ is smoothed out in the manner discussed by Hall and Lakin (1988). However for simplicity we will assume below that there is just one such location in the flow, later we will simply sum over all such locations.

We perturb α, c, y_d, and P by writing

$$\begin{aligned} \alpha &= \overline{\alpha} + \tilde{\alpha}, \\ c &= \overline{c} + \tilde{c}, \\ y_d &= \overline{y}_d + \tilde{y}_d, \\ P &= \overline{P} + \tilde{P}. \end{aligned} \tag{2.23a, b, c}$$

We assume that the quantities denoted by a tilde are smaller than those with a bar. The equations satisfied by $\overline{P}, \tilde{P}$ are

$$\left.\begin{aligned} &(\overline{u} - \overline{c})(\overline{P}'' - \overline{\alpha}^2 \overline{P}) - 2\overline{u}'\overline{P}' = 0, \\ &\overline{P}_y = 0, \ y = 0, \ \overline{P} \to 0, \ y \to \infty, \end{aligned}\right\} \tag{2.24}$$

and

$$\left.(\overline{u} - \overline{c})(\tilde{P}'' - \overline{\alpha}^2 \tilde{P}) - 2\overline{u}'\tilde{P}' = (\tilde{u} - \tilde{c})(\overline{P}'' - \overline{\alpha}^2 \overline{P}) + 2\tilde{\alpha}\overline{\alpha}(\overline{u} - \overline{c})\overline{P}\right\} \tag{2.25}$$

$$\tilde{P}_y = 0, \ y = 0, \ \tilde{P} \to 0, \ y \to \infty.$$

The system (2.24) constitutes an eigenvalue problem for $\overline{\alpha}$ with $\overline{c} = \overline{u}(y_c)$ with y_c defined by $\overline{u}''(y_c) = 0$. The second system only has a solution if the appropriate orthogonality condition is satisfied. However we confine our attention to constant frequency neutral modes so that $\tilde{c}$ is given by

$$\tilde{c} = -\frac{-\overline{u}_c' \tilde{u}_c''}{\overline{u}_c'''} + \tilde{u}_c \tag{2.26}$$

Here a subscript c denotes a quantity evaluated at the critical layer. The wavenumber perturbation $\tilde{\alpha}$ is then determined by

$$\tilde{\alpha}\bar{c} + \tilde{c}\bar{\alpha} = 0.$$

Perhaps the easiest way to determine $\tilde{\alpha}$ is to integrate the differential equation for $\tilde{P}$ once by writing $\tilde{P} = \overline{P}F(y)$. In order to write down the condition that the solution obtained in this way is continuous across $y = y_c$ we first assume that in the neighborhood of y_c, with $\tilde{y} = y - y_c$,

$$\begin{aligned} \tilde{u} &= \phi_0 + \tilde{y}\phi_1 + \tilde{y}^2\phi_2 + \cdots \\ \overline{u} &= \overline{c} + \tilde{y}\mu_1 + \frac{\tilde{y}^2}{2}\mu_2 + \cdots, \ \mu_2 = 0, \end{aligned} \tag{2.27a, b}$$

$$\overline{P} = 1 + \frac{\tilde{y}^2 Q_2}{2!} + \frac{\tilde{y}^3 Q_3}{3!} + \cdots, \ Q_2 == -\overline{\alpha}^2.$$

Here we have normalized $\overline{P}$ to unity at the critical layer. After some manipulation we find that the condition to determine $\tilde{\alpha}$ is

$$\int_0^\infty [\frac{\overline{P}\left\{2\tilde{\alpha}\overline{\alpha}(\overline{u}-\overline{c})\overline{P} + 2\tilde{u}'\overline{P}' - \frac{2\overline{u}'(\tilde{u}-\tilde{c})\overline{P}'}{\overline{u}-\overline{c}}\right\}}{(\overline{u}-\overline{c})^3} - \frac{C}{\tilde{y}^3} - \frac{D}{\tilde{y}^2}]dy = \frac{-C}{2y_c^2} + \frac{D}{y_c} + E \tag{2.28}$$

where C, D, E are defined by

$$\begin{aligned} C &= 2Q_2\mu_1^{-3}[\tilde{c} - \phi_0] \\ D &= \mu_1^{-3}\left[Q_3(\tilde{c} - \phi_0) - \frac{2\overline{\alpha}^2\mu_1\tilde{c}}{\overline{c}}\right]. \\ E &= \tilde{y}_d^2\bar{P}_d\bar{P}_d'[\bar{u}''(y_d^+) - \bar{u}''(y_d^-)][\bar{u}(y_d) - \bar{c}]^{-3} \end{aligned} \tag{2.29a, b}$$

Here we have denoted quantities evaluated at $\tilde{y}_d$ by a d subscript and $[\bar{u}''(y_d^+) - \bar{u}''(y_d^-)]$ denotes the jump in $\bar{u}''$ across $\tilde{y}_d$. For a given profile $\tilde{u}$ we can compute $\tilde{\alpha}$ from (2.28) with $\tilde{c}$ given by (2.26), however in the vortex-wave interaction we have to maintain a constant frequency for the wave as it moves downstream. Thus if $\tilde{u}$ represents the change in u over some small distance in x then $\tilde{\alpha}$ determined in this way must satisfy

$$\tilde{\alpha}\overline{c} + \overline{\alpha}\tilde{c} = 0 \tag{2.30}$$

and in general $\tilde{\alpha}, \tilde{c}$ determined by (2.26), (2.28) will not satisfy this condition. Thus the role of the wave forcing the mean state now becomes apparent since $B(x)$, the Rayleigh wave amplitude, must now adjust in order to enable us to satisfy this condition (2.30). Now we shall investigate the possible growth of small but finite amplitude Rayleigh waves from a strongly nonlinear vortex flow.

3. Small Amplitude Rayleigh Waves Bifurcating from Strongly Nonlinear Vortex Flows

In the first instance we shall seek finite amplitude solutions of the interaction equations (2.16) - (2.22) appropriate to the situation when the Rayleigh wave is small. The expansions we develop are related to those given by Hall and Lakin (1988) for the case when no Rayleigh forcing occurs. The major difference between the two expansion procedures is that the initial vortex in the work of Hall and Lakin was calculated from the limiting form of the weakly nonlinear theory of Hall (1982b). Before discussing how we must incorporate the Rayleigh wave into the expansions procedure we comment on the solution of (2.16) - (2.22) when $B = 0$.

Suppose then that at the initial location, taken without loss of generality to be $x = 1$, we have

$$u = \bar{u}(y).$$

In the absence of any vortex activity the mean flow function u can be expanded in powers of $X = x - 1$, we must however assume that $\bar{u}'''(0) = 0$ in order that no logarithmic terms occur in the expansions. At first sight we would expect that the expansions of y_1 and y_2 should proceed in powers of X. However, unless certain quite severe restrictions are imposed on $\bar{u}$, it can be shown that the required continuity and boundary conditions cannot be satisfied. This occurs because the existence of a vortex $X = 0$ means that $\bar{u}, \bar{u}_y$ are initially continuous at y_1, y_2. Thus the contributions to the matching conditions on $\bar{u}$ a small distance beyond $X = 0$ arising from the perturbations to y_1, y_2 vanish at $0(X)$. In effect we must therefore allow y_1 and y_2 to be perturbed by $0(X^{1/2})$ so that the second order, ie $0(X)$, correction terms come into play. We therefore write

$$\begin{aligned} y_1 &= \bar{y}_1 + X^{1/2}\bar{y}_{11} + X\bar{y}_{12} + \cdots, \\ y_2 &= \bar{y}_2 + X^{1/2}\bar{y}_{21} + X\bar{y}_{22} + \cdots, \end{aligned} \qquad (3.1a, b)$$

and note that the special case $\bar{y}_1 = \bar{y}_2$ is discussed in the next section. We anticipate a similar expansion for the Rayleigh disturbance amplitude whilst the wavenumber and wavespeed expand in whole powers of X. We therefore write

$$\begin{aligned} B &= X^{1/2}\bar{B} + X\bar{B}_1 + \cdots, \\ \alpha &= \bar{\alpha} + X\tilde{\alpha} + \cdots, \\ c &= \bar{c} + X\tilde{c} + \cdots, \end{aligned} \qquad (3.2a, b, c)$$

with

$$\tilde{\alpha}\bar{c} + \bar{\alpha}\tilde{c} = 0, \qquad (3.3)$$

if the Rayleigh wave is to remain neutral and of constant frequency as it moves downstream. In Regions I, III we write

$$\begin{aligned} u &= u_B = \bar{u} + X\bar{u}_1 + X^2\bar{u}_2 + \cdots, \\ v &= v_B = \bar{v} + X\bar{v}_1 + X^2\bar{v}_2 + \cdots, \\ \bar{p}' &= \pi_0 + X\pi_1 + X^2\pi_2 + \cdots. \end{aligned} \qquad (3.4a, b, c)$$

whilst the wall curvature expands as

$$\chi = \chi_0 + X\chi_1 + X^2\chi_2 + \cdots .$$

If we substitute the above expansions into (2.16) and equate terms of $O(X^0)$ we obtain

$$\begin{aligned} \bar{u}_1 + \bar{v}_y &= 0, \\ \bar{u}\bar{u}_1 + \bar{v}\bar{u}_y &= -\pi_0 + \bar{u}_{yy}, \end{aligned} \qquad (3.5a, b)$$

so that in Regions I, III the required solutions are

$$\begin{aligned} \bar{u}_1 &= \frac{\bar{u}'' - \pi_0}{\bar{u}} - \bar{u}' \left\{ J_j + \int_{\bar{y}_j}^{y} (\pi_0 - \bar{u}'')\bar{u}^{-2} dy \right\}, \\ \bar{v} &= J_j \bar{u} + \bar{u} \int_{\bar{y}_j}^{y} [\pi_0 - \bar{u}'']\bar{u}^{-2} dy, \end{aligned} \qquad (3.6a, b)$$

with $j = 1, 2$ and J_1, J_2 constants to be determined.

In fact J_1 is fixed by the condition $\bar{u}_1 = 0,\ y = 0$, this gives

$$J_1 = \int_0^{\bar{y}_1} (\pi_0 - \bar{u}'')\bar{u}^{-2} dy. \qquad (3.7)$$

and we note that the above integral exists if we assume that $\bar{u}'' = \pi_0,\ \bar{u}''' = 0,\ y = 0$. In the core region III we retain (3.4) and expand the Rayleigh pressure function P as

$$P = \bar{P} + X\bar{P}_1 + \cdots .$$

At zeroth order the coreflow downstream velocity component is given by

$$\bar{u} = \frac{\sqrt{A + 2y}}{\sqrt{\chi_0 G}},$$

and the zeroth order approximations to (2.17a,b) yield

$$\bar{u}_1 = \frac{1}{\sqrt{\chi_0 G}\sqrt{A + 2y}} \left\{ K - \bar{B}^2 I(y) \right\} - \frac{\chi_1 \sqrt{A + 2y}}{2\chi_0 \sqrt{\chi_0 G}},$$

$$\bar{v} = b - \frac{\sqrt{A + 2y}}{\sqrt{\chi_0 G}} \left\{ K - \bar{B}^2 I(y) \right\} - \frac{\bar{B}^2}{\sqrt{\chi_0 G}} \int_{\bar{y}_1}^{y} \sqrt{A + 2y}\, I' dy + \frac{\chi_1 (A + 2y)^{\frac{3}{2}}}{6\chi_0 (\chi_0 G)^{\frac{1}{2}}},$$

where

$$I(y) = -(\chi_0 G)^{3/2} \int_{\bar{y}_1}^{y} \frac{\left\{ \bar{P}'^2 (\sqrt{A + 2y} - \bar{c}\sqrt{\chi_0 G})^{-2} \right\}' dy}{(\sqrt{A + 2y} - \bar{c}(\chi_0 G)^{1/2})}. \qquad (3.8a, b, c)$$

Here the constants A, B, K and b are to be determined. In fact the zeroth order approximations to the condition that $\bar{v}$ is continuous at $\bar{y}_j$ yield

$$\begin{aligned} \sqrt{A + 2\bar{y}_1}(J_1 + K) &= b\sqrt{G\chi_0} + \tfrac{\chi_1}{6\chi_0}(A + 2\bar{y}_1)^{\frac{3}{2}}, \\ \sqrt{A + 2\bar{y}_2}(J_2 + K - \bar{B}^2 I_2) &= b\sqrt{G\chi_0} - \bar{B}^2 \int_{\bar{y}_1}^{\bar{y}_2} \sqrt{A + 2y}\, I' dy + \frac{\chi_1}{6\chi_0}(A + 2\bar{y}_2)^{\frac{3}{2}}. \end{aligned} \qquad (3.9a, b)$$

In addition the continuity of u, u_y at zeroth order at $y = y_1, y_2$ yields

$$\frac{\sqrt{A+2\bar{y}_j}}{\sqrt{\chi_0 G}} = \bar{u}(\bar{y}_j),\ \ \bar{u}'(\bar{y}_j) = \frac{1}{\chi_0 G\bar{u}(\bar{y}_j)}. \tag{3.10a, b}$$

We recall that J_1 must satisfy (3.7) in order that $\bar{u}_1(0) = 0$, then (3.9a,b) determine b and J_2 in terms of one remaining unknown constant K. We now use the coreflow expansion to show that the zeroth order approximation to the vortex condition (2.18) is

$$\sqrt{\chi_0 G}\left[-\frac{b\sqrt{A+2y}}{\sqrt{\chi_0 G}} + \frac{1}{12}\frac{(A+2y)^2\chi_1}{\chi_0^2 G} + \frac{1}{\sqrt{A+2y}\sqrt{\chi_0 G}} - \pi_0 y\right]_{\bar{y}_1}^{\bar{y}_2} = \overline{B}^2\hat{I}(S_2 - S_1).$$

where

$$S_j = \sqrt{A+2\bar{y}_j}.$$

and

$$\hat{I}(S_2 - S_1) = \frac{1}{\sqrt{\chi_0 G}}\int_{\bar{y}_1}^{\bar{y}_2} I(y)(\sqrt{A+2y_2}/\sqrt{A+2y} - 2)dy.$$

Hence the above equation together with (3.7), (3.9a,b) yield

$$\begin{aligned}
b &= b_1 - \overline{B}^2\hat{I},\\
b_1 &= \frac{\sqrt{\chi_0 G}}{S_2 - S_1}\left\{\frac{\chi_1}{12\chi_0^2 G}\left[S_2^4 - S_1^4\right] + \frac{1}{\sqrt{\chi_0 G}}[S_2^{-1} - S_1^{-1}] - \pi_0(\bar{y}_2 - \bar{y}_1)\right\},\\
K &= \frac{b\sqrt{\chi_0 G}}{S_1} - \int_0^{\bar{y}_1}(\pi_0 - \bar{u}'')\bar{u}^{-2}dy + \frac{\chi_1 S_1^2}{6\chi_0}, \qquad (3.11a, b, c)\\
J_2 &= -K + \bar{B}^2 I_2 + \frac{b\sqrt{\chi_0 G}}{S_2} - \frac{\bar{B}^2\int_{\bar{y}_1}^{\bar{y}_2}\sqrt{A+2y}I'dy}{S_2} + \frac{\chi_1 S_2^2}{6\chi_0}.
\end{aligned}$$

We now substitute for $\bar{u}_1$ from (3.6a) into (3.4a) and expand about $y = \bar{y}_1, \bar{y}_2$ to obtain

$$u = \bar{u}^j + X^{1/2}\bar{u}'^j\bar{y}_{j1} + X\left\{\bar{u}'^j\bar{y}_{j2} + \bar{u}''^j\frac{\bar{y}_{j1}^2}{2} + \frac{\bar{u}''^j - \pi_0}{\bar{u}^j} - \bar{u}'^j J_j\right\} + \cdots,$$

$$u_y = \bar{u}'^j + X^{1/2}\bar{u}''^j\bar{y}_{j1} + X\left\{\bar{u}''^j\bar{y}_{j2} + \bar{u}'''^j\frac{\bar{y}_{j1}^2}{2} + \left(\frac{\bar{u}'' - \pi_0}{\bar{u}}\right)'^j - \frac{\bar{u}'^j}{\bar{u}^{j2}}[\pi_0 - \bar{u}''^j] - J_j\bar{u}''^j\right\} + \cdots. \tag{3.12a, b}$$

Here $\bar{u}'^j, \bar{u}''^j$, etc. denote $\bar{u}'(\bar{y}_j),\ \bar{u}''(\bar{y}_j)$, etc. and the next order correction terms are $O(X^{3/2})$. If we perform similar expansions for the coreflow solution we obtain

$$\begin{aligned}
u &= \frac{S_j}{\sqrt{\chi_0 G}} + \frac{X^{1/2}\bar{y}_{j1}}{\sqrt{\chi_0 G}S_j} + X\left\{\frac{1}{\sqrt{\chi_0 G}S_j}[K + \bar{y}_{j2} - \bar{B}^2 I_j] - \frac{\chi_1 S_j}{2\chi_0\sqrt{\chi_0 G}} - \frac{1}{2\sqrt{\chi_0 G}}\frac{\bar{y}_{j1}^2}{S_j^3}\right\} + \cdots,\\
u_y &= \frac{1}{\sqrt{\chi_0 G}S_j} - \frac{X^{1/2}\bar{y}_{j1}}{\sqrt{\chi_0 G}S_j^3} + \cdots,
\end{aligned} \tag{3.12c, d}$$

If the initial profile is chosen such that $\bar{u}^j = S_j/\sqrt{\chi_0 G}$, $\bar{u}'^j = 1/S_j\sqrt{\chi_0 G})$ we see that the order $X^0, X^{1/2}$, terms in (3.12a) and (3.12c) are consistent whilst also are the order X^0 terms in (3.12b), (3.12d). However if we are to make u and u_y consistent up to orders $X, X^{1/2}$ respectively then we have four equations to satisfy but only three unknowns $\bar{y}_{11}$, $\bar{y}_{21}$, $\bar{B}^2$ at our disposal. Thus we cannot in general make u, u_y consistent at these orders and the only possible remedy is to allow for diffusion layers of depth $X^{1/2}$ in the neighborhood of $\bar{y}_1, \bar{y}_2$. However these layers occur only in I, II where the mean flow satisfies the boundary layer equations. These blending layers occur where $\partial_y^2 \sim \partial_x$ and enable jumps in u, u_y across the layers to be accommodated.

We shall only consider the behavior near $y = \bar{y}_2$, a similar structure holds at $y = \bar{y}_1$. We define a variable ξ by

$$\xi = [y - \bar{y}_2]X^{-1/2}\left(\frac{S_2}{2\sqrt{\chi_0 G}}\right)^{1/2}$$

so that $\frac{\partial}{\partial x} \to \frac{\partial}{\partial X} - \frac{\xi}{2X}\frac{\partial}{\partial \xi}$. We modify (3.4a,b) near $y = y_2$ by writing

$$\begin{aligned} u &= u_B + X\hat{U}(\xi) + \cdots, \\ v &= v_B + X^{1/2}\hat{V}(\xi) + \cdots, \end{aligned} \qquad (3.13a,b)$$

and it is straightforward to show that $\hat{U}, \hat{V}$ satisfy

$$\begin{aligned} \hat{U}'' &= 2\left(\hat{U} - \tfrac{\xi}{2}\hat{U}'\right), \\ \left(\tfrac{S_2}{2\sqrt{\chi_0 G}}\right)^{1/2}\hat{V}' &= \left(\tfrac{\xi}{2}\hat{U}' - \hat{U}\right). \end{aligned} \qquad (3.14c,d)$$

The solution of (3.14c) which decays when $\xi \to \infty$ is

$$\hat{U}e^{-\xi_2^2/4} = C_2 e^{-\xi^2/4}U\left(\frac{5}{2}, \xi\right), \qquad (3.15)$$

where $U(a,x)$ denotes the parabolic cylinder function, C_2 is a constant and $\xi_2 = \bar{y}_{21}\left(\frac{S_2}{2\sqrt{\chi_0 G}}\right)^{1/2}$.

The effect of this extra term in the expansion of u, u_y near y_2 is to produce terms $C_2 X U\left(\frac{5}{2}, \xi_2\right), -\, C_2 X^{1/2}\left(\frac{S_2}{2\sqrt{\chi_0 G}}\right)^{1/2} U\left(\frac{3}{2}, \xi_2\right)$ on the right hand sides of (3.12a,b) for $j = 2$. A similar analyses near $y = \bar{y}_1$ produces similar terms. We are now in a position to make (3.11a), (3.12a) consistent up to order X and (3.11b), (3.12b) consistent up to order $X^{1/2}$. This is achieved if

$$\begin{aligned} \left\{\bar{u}''^j\frac{\bar{y}_{j1}^2}{2} + \frac{\bar{u}_j'' - \pi_0}{\bar{u}^j} - \bar{u}'^j J_j\right\} + C_j U\left(\frac{5}{2}, (-1)^j\xi_j\right) &= \frac{1}{\sqrt{\chi_0 G}S_j}[K - \bar{B}^2 I_j] - \frac{\chi_1 S_j}{2\chi_0\sqrt{\chi_0 G}} \\ &\quad - \frac{1}{2\sqrt{\chi_0 G}}\frac{\bar{y}_{j1}^2}{S_j^3}, \\ \bar{u}''^j\bar{y}_{j1} + (-1)^{j+1}\left(\frac{S_j}{2\sqrt{\chi_0 G}}\right)^{1/2} C_j U\left(\frac{3}{2}, (-1)^j\xi_j\right) &= \frac{-\bar{y}_{j1}}{\sqrt{\chi_0 G}S_j^3}. \end{aligned} \qquad (3.16a,b)$$

If C_1, C_2 are eliminated above we obtain

$$\Lambda_j\left\{\overline{y}_{j1}^2+\omega_j\overline{y}_{j1}\right\}=\frac{1}{\sqrt{\chi_0 G S_j}}[K-\overline{B}^2 I_j]-\frac{\chi_1 S_j}{2\chi_0\sqrt{\chi_0 G}}-\frac{(\overline{u}''^j-\pi_0)}{\overline{u}^j}+\overline{u}'^j J_j \qquad (3.17a)$$

with

$$\Lambda_j=\frac{\bar{u}''^j}{2}+\frac{1}{2\sqrt{\chi_0 G S_j^3}}, \quad \omega_j=2^{\frac{3}{2}}(-1)^j\frac{U\left(\frac{5}{2},(-1)^j\xi_j\right)}{U\left(\frac{3}{2},(-1)^j\xi_j\right)}\frac{(\chi_0 G)^{\frac{1}{4}}}{\sqrt{S_j}}. \qquad (3.17b,c)$$

In the absence of a Rayleigh wave, ie when $\bar{B}=0$, we can substitute for K from (3.12b) with b given by (3.12a) to give a quadratic equation for $\bar{y}_{j1}$. In the presence of a Rayleigh wave we can only express $\bar{y}_{j1}$ in terms of $\bar{B}^2$. However if we now derive the condition that the wave remains neutral up to $O(X)$ then a further equation linking $\bar{y}_{j1}, \bar{B}^2$ will be obtained. In addition we note that if the initial data is such that Λ_j vanishes for $j=1,2$ then the expansions for y_j proceed in whole powers of X.

If we use the result (2.29) applied to the $O(X)$ correction to the mean velocity field we find that

$$h_1+Kh_2+\bar{B}^2h_3+J_2h_4+C_1h_5U(\frac{5}{2},(-1)^j\xi_1)+C_2h_6U(\frac{5}{2},(-1)^j\xi_2)=0. \qquad (3.18)$$

Here the constants $h_1, h_2, h_3, h_4, h_5, h_6$ are defined by

$$\begin{aligned}
h_1 \;=\;& \int_0^\infty\left[\bar{P}\left\{\frac{2\tilde{\alpha}\tilde{\alpha}\bar{P}}{(\bar{u}-\bar{c})^2}+\frac{2\bar{u}'\tilde{c}\bar{P}'}{(\bar{u}-\bar{c})^4}\right\}-\frac{C}{(y-y_c)^3}-\frac{D}{(y-y_c)^2}\right]dy+\frac{C}{2y_c^2}-\frac{D}{y_c}\\
&+\int_0^{\bar{y}_1}\frac{2\bar{P}\bar{P}'}{(\bar{u}-\bar{c})^3}\left\{\mathcal{N}'-\frac{\bar{u}'\mathcal{N}}{\bar{u}-\bar{c}}\right\}dy+\chi_1\int_{\bar{y}_1}^{\bar{y}_2}\frac{\bar{P}\bar{P}'}{(\bar{u}-\bar{c})^4}\frac{(\bar{c}-2\bar{u})}{\chi_0^2 G\bar{u}}dy\\
&+\bar{y}_{11^2}(\bar{P}_1^2)'\Lambda_1[\bar{u}(\bar{y}_1)-\bar{c}]^{-3}-\bar{y}_{21^2}(\bar{P}_2^2)'\Lambda_2[\bar{u}(\bar{y}_2)-\bar{c}]^{-3}\\
&+\int_{\bar{y}_2}^\infty\frac{2\bar{P}\bar{P}'}{(\bar{u}-\bar{c})^3}\left(\mathcal{M}'-\frac{\bar{u}'\mathcal{M}}{\bar{u}-\bar{c}}\right)dy,
\end{aligned}$$

$$h_2=\int_{\bar{y}_1}^{\bar{y}_2}\frac{2\bar{P}\bar{P}'}{(\bar{u}-\bar{c})^4}\left\{\chi_0^2G\bar{u}\right\}^{-2}[\frac{\bar{c}}{\bar{u}}-2]dy,$$

$$h_3=-\int_{\bar{y}_1}^{\bar{y}_2}\frac{2\bar{P}\bar{P}'}{(\bar{u}-\bar{c})^3}\left\{\frac{I[\bar{c}-2\bar{u}]}{\bar{u}^3\chi_0^2G^2(\bar{u}-\bar{c})}+\frac{I'}{\chi_0 G\bar{u}}\right\}dy,$$

$$h_4=-\int_{\bar{y}_2}^\infty\frac{2\bar{P}\bar{P}'}{(\bar{u}-\bar{c})^3}\left\{\bar{u}''-\frac{\bar{u}'^2}{\bar{u}-\bar{c}}\right\}dy,$$

$$h_5=\frac{2\bar{P}_1\bar{P}_1'}{(\bar{u}(\bar{y}_1)-\bar{c})^3},$$

$$h_6=-\frac{2\bar{P}_2\bar{P}_2'}{(\bar{u}(\bar{y}_2)-\bar{c})^3},$$

with

$$\mathcal{N}(y) = \frac{\bar{u}'' - \pi_0}{\bar{u}} - \bar{u}' \int_0^y \left(\frac{\pi_0 - \bar{u}''}{\bar{u}^2} \right) dy,$$

$$\mathcal{M}(y) = \frac{\bar{u}'' - \pi_0}{\bar{u}} - \bar{u}' \int_{\bar{y}_2}^y \left(\frac{\pi_0 - \bar{u}''}{\bar{u}^2} \right) dy.$$

and C, D given by (2.29) with $\tilde{c} = \bar{u}_{1c} - \frac{\bar{u}_c' \bar{u}_{1c}''}{\bar{u}_c'''}$, and $\bar{u}_1$ given by (3.6a). Furthermore $\bar{P}_j, \bar{P}_j'$ etc. denote $\bar{P}, \bar{P}'$ evaluated at $\bar{y}_j$. If we eliminate C_1, C_2 and J_2 from (3.18) using (3.16b) and (3.12c) we obtain

$$h_7 \bar{B}^2 = h_8 + h_5 \omega_1 \Lambda_1 \bar{y}_{11} + h_6 \omega_2 \Lambda_2 \bar{y}_{21} \tag{3.19}$$

with

$$h_7 = -\hat{I}\sqrt{\chi_0 G}\left\{\frac{\hat{h}_2}{S_1} + \frac{h_4}{S_2}\right\} + h_3 + h_4\left\{I_2 - \int_{\bar{y}_1}^{\bar{y}_2} \frac{\sqrt{A + 2y} I' dy}{S_2}\right\}$$

$$h_8 = -h_1 - \hat{h}_2\left\{\frac{b_1\sqrt{\chi_0 G}}{S_1} + \frac{\chi_1 S_1^2}{6\chi_0} - J_1\right\} - h_4\left[\frac{b_1\sqrt{\chi_0 G}}{S_2} + \frac{\chi_1 S_2^2}{6\chi_0}\right]$$

$$\hat{h}_2 = h_2 - h_4.$$

Finally we then substitute for $\bar{B}^2$ from (3.19) into (3.17a) with $j = 1, 2$ to give the coupled equations for $\bar{y}_{11}, \bar{y}_{21}$:

$$\begin{aligned} \Lambda_1\left\{\bar{y}_{11}^2 + \bar{y}_{11}\omega_1\left(1 - \frac{h_9 h_5}{h_7}\right)\right\} &= h_{10} + h_9 h_8 h_7^{-1} + \omega_2 \Lambda_2 h_6 h_9 h_7^{-1} \bar{y}_{21}, \\ \Lambda_2\left\{\bar{y}_{21}^2 + \bar{y}_{21}\omega_2\left(1 - \frac{h_{11} h_6}{h_7}\right)\right\} &= h_{12} + h_{11} h_8 h_7^{-1} + \omega_1 \Lambda_1 h_5 h_{11} h_7^{-1} \bar{y}_{11}, \end{aligned} \tag{3.20a, b}$$

with

$$\begin{aligned} h_9 &= -\frac{\hat{I}}{S_1^2}, & h_{10} &= \frac{b_1}{S_1^2} - \frac{\chi_1 S_1}{3\chi_0\sqrt{\chi_0 G}} - \frac{(\bar{u}''^1 - \pi_0)}{\bar{u}^1} \\ h_{11} &= -\frac{\hat{I}}{S_2^2} - \frac{I^+}{\sqrt{\chi_0 G} S_2^2}, & h_{12} &= \frac{b_1}{S_2^2} - \frac{\chi_1 S_2}{3\chi_0\sqrt{\chi_0 G}} - \frac{(\bar{u}''^2 - \pi_0)}{\bar{u}^2} \end{aligned} \tag{3.21a, b, c, d}$$

and I^+defined by

$$I^+ = \int_{\bar{y}_1}^{\bar{y}_2} I'\sqrt{A + 2y}\, dy$$

Thus the initial small amplitude form of the vortex and wave are determined at leading order when the nonlinear equations (3.20a,b) are solved for $\bar{y}_{11}, \bar{y}_{21}$. Note however that h_8 appearing above is a quadratic function of these quantities whilst ω_1, ω_2 depend on $\bar{y}_{11}\bar{y}_{21}$, through parabolic cylinder functions so that the solution of (3.20) must be found numerically.

A numerical example

In order to proceed further we must specify an initial velocity profile and curvature distribution. Without any loss of generality we may take $\chi_0 = 1$ and we then take the initial profile $\bar{u}$ to be given by

$$\bar{u} = \left\{\lambda y + \frac{1}{8}(y^2 - \frac{2y^4}{3})\right\}, \quad 0 < y < \bar{y}_1 = 1$$

$$\bar{u} = \frac{\sqrt{\bar{A} + 2y}}{\sqrt{G}}, \quad \bar{y}_1 < y < \bar{y}_2 = 2,$$

$$\bar{u} = \frac{\sqrt{\bar{A} + 2\bar{y}_2}}{\sqrt{G}} + 1 - \exp\left\{\frac{\bar{y} - y}{\sqrt{\bar{A} + 2\bar{y}_2}\sqrt{G}}\right\}, \quad \bar{y}_2 < y < \infty \qquad (3.22a, b, c)$$

Here the constant λ is to be specified but we restrict our attention to situations when $\bar{u}'$ is always positive. In order that $\bar{u}$ and $\bar{u}_y$ are continuous at the positions $y = \bar{y}_1, \bar{y}_2$ we require that A and G are defined by

$$A = \frac{-(\lambda + \frac{5}{6})}{(\lambda + \frac{2}{3})},$$

$$G = \frac{1}{(\lambda + \frac{5}{6})(\lambda + \frac{2}{3})}.$$

The profile defined above has an inflection point at $y = \frac{1}{2}$ so that the wavespeed is given by

$$\bar{c} = \frac{\lambda}{2} + \frac{5}{192}. \qquad (3.23)$$

The first step in the calculations is to solve Rayleigh's equation for $\bar{P}$ with $\bar{u}$ and $\bar{c}$ as given by (3.22) and (3.23). In Figure (3.1) we show the wavenumber $\bar{\alpha}$ as a function of λ. The eigenfunctions $\bar{P}$ associated with these modes are shown in Figure (3.2). Though it cannot be seen easily in this figure we note that at the transition layers y_1, y_2 the quantities $\bar{P}, \bar{P}'$ are continuous whilst $\bar{P}''$ is discontinuous.

The constants h_1, h_2, etc appearing in (3.21) can then be determined using Simpsons Rule to evaluate the integral and a routine to evaluate parabolic cylinder functions. This was done using either the series or large argument asymptotic form of the parabolic cylinder functions dependent on the size of the argument. The equations (3.21) were then solved using a Newton iteration procedure. The calculations were restricted to the case $\lambda = .5, 1, 1.5, 2, 2.5, 3.$ but different values of the curvature parameter χ_1 were used. In Figures (3.3a,b,c) we show the computed values of $\bar{y}_1$, $\bar{y}_2$ and $\bar{B}^2$ as functions of χ_1. The calculations were carried out for the values of λ used to calculate Figures (3.1,3.2) but in fact we were unable to find solutions of (3.21) for the case $\lambda = .5$. Furthermore the solutions shown could not be extended beyond the ranges of χ_1 shown. At the lower end of the ranges in question the results showed that $\bar{y}_{11}, \bar{y}_{21}$ were tending to a constant plus a multiple of the square root of the difference between χ_1 and it's cutoff value. This suggested to us that at such a point two solutions of (3.21) were coalescing, however careful searches were unable to find a second root. However it could well be that other roots exist but our Newton iteration procedure was not able to detect them. Since the main aim of the present calculation was to determine whether physically acceptable solutions of the vortex-wave interaction equations could be found near the position of neutral instability of a Rayleigh wave we did not investigate the matter further.

In fact an inspection of Figure (3.3c) shows that $\bar{B}^2$ is negative for a range of values of χ_1. These solutions are not physically acceptable since $\bar{B}$ must be real. At first sight it might appear that these solutions are relevant to the case when the Rayleigh wave is bifurcating subcritically from $x = 1$. This is not the case since our expansion procedure cannot be simply modified to take account of $x - 1$ being negative. This is because the blending layer structure now has solutions increasing exponentially away from y_1, y_2 so that the matching of u, u_y, v cannot be achieved where I, II and II, III meet. Nevertheless we feel that the solutions of (3,21) with negative $\bar{B}^2$ do have some physical significance; we return to that point in the final section of this paper.

4. The Spontaneous Generation of Rayleigh Waves/Vortices

Suppose that we have an incoming inflectional velocity profile $u = \bar{u}(y)$ at $x = 1$. We use the notation of the previous section and denote the critical level by $y = y_c$ and denote quantities evaluated at the critical layer by a subscript c. We suppose further that the position $y = \bar{y}$ at which $G\bar{u}\bar{u}_y = 1$, where finite amplitude vortices emerge, is, without any loss of generality, such that $\bar{y} > y_c > 1$.

In the first instance we ignore the mean flow correction driven by the vortex structure and write

$$\begin{aligned} u &= \bar{u} + X\bar{u}_1 + X^2\bar{u}_2 + \cdots, \\ v &= \bar{v} + X\bar{v}_1 + X^2\bar{v}_2 + \cdots, \\ \bar{p}' &= \pi_0 + X\pi_1 + X^2\pi_2 + \cdots, \end{aligned} \qquad (4.1a, b, c)$$

where $X = x - 1$. The expansions for u, v fail when $y = O(X^{1/3})$ where a passive boundary layer is needed to satisfy the no-slip condition. It should be noted here that $\bar{v}$ cannot be specified arbitrarily but must be determined in terms of the initial streamwise profile $\bar{u}(y)$. If the above expansions are substituted into the boundary layer equations then we can show that

$$\bar{u}_1 + \bar{v}' = 0, \quad \bar{u}\bar{u}_1 + \bar{v}\bar{u}' = -\pi_0 + \bar{u}'', \qquad (4.2a, b)$$

and

$$2\bar{u}_2 + \bar{v}_1' = 0, \quad 2\bar{u}\bar{u}_2 + \bar{u}_1^2 + \bar{v}\bar{u}_1' + \bar{v}_1\bar{u}' = -\pi_1 + \bar{u}_1''. \qquad (4.3a, b)$$

These equations can be integrated to give

$$\begin{aligned} \bar{v} &= \bar{u}\int_0^y [\pi_0 - \bar{u}'']\bar{u}^{-2}dy, \\ \bar{u}_1 &= -\bar{u}'\int_0^y [\pi_0 - \bar{u}'']\bar{u}^{-2}dy + \left(\frac{\pi_0 - \bar{u}''}{\bar{u}}\right\}, \end{aligned} \qquad (4.4a, b)$$

and

$$\bar{v}_1 = \bar{u}\int_0^y [\pi_1 - \bar{u}_1'' + \bar{u}_1^2 + \bar{v}\bar{u}_1']\bar{u}^{-2}dy,$$
$$\bar{u}_2 = -\bar{u}'\int_0^y [\pi_1 + \bar{u}_1^2 - \bar{u}_1'' + \bar{v}\bar{u}_1']\bar{u}^{-2}dy + \frac{\{\pi_1 - \bar{u}_1'' + \bar{u}_1^2 + \bar{v}\bar{u}_1'\}}{\bar{u}}. \qquad (4.4a,b)$$

Here we have made assumptions about the behavior of $\bar{u}$ near $y = 0$ in order that the integrals shown exist at $y = 0$. If these assumptions are not made then the solution for $y = 0(1)$ must have terms involving $\log X$ in its small X form. In view of the discussion given at the end of the section 2 it follows that α and c for the Rayleigh wave associated with the perturbed velocity profile (4.1) must expand as

$$\alpha = \bar{\alpha} + \tilde{\alpha}_1 X + \tilde{\alpha}_2 X^2 + \cdots,$$
$$c = \bar{c} + \tilde{c}_1 X + \tilde{c}_2 X^2 + \cdots . \qquad (4.5a,b)$$

If we use the notation of §2 then, in the absence of vortex activity, it follows that $\tilde{\alpha}_1, \tilde{c}_1, \tilde{\alpha}_2, \tilde{c}_2$ are determined by

$$\int_0^\infty \bar{P}\frac{\left\{2\tilde{\alpha}_j\bar{\alpha}(\bar{u}-\bar{c})\bar{P} + 2\bar{u}'\frac{(\bar{u}_j-\tilde{c}_j)}{(\bar{u}-\bar{c})}\bar{P}' + 2\bar{u}'\bar{P}' - \frac{C_j}{\bar{y}^3} - \frac{D_j}{\bar{y}^2}\right\}}{(\bar{u}-\bar{c})^2}dy = -\frac{C_j}{2y_c^2} + \frac{D_j}{y_c}, \quad j = 1,2. \quad (4.6)$$

Here C_j, D_j satisfy (2.29) with $(\tilde{c} - \phi_0)$ replaced by $(\tilde{c}_j - \bar{u}_j(y_c))$. However if the wave is to remain neutral and of constant frequency we also require that

$$\tilde{c}_1 = \frac{-\bar{u}_c'\bar{u}_{1c}''}{\bar{u}_c'''} + \bar{u}_{1c}$$
$$\tilde{c}_2 = \frac{\bar{u}_c'}{\bar{u}_c'''}\left\{\frac{\bar{u}_{1c}''\bar{u}_{1c}'''}{\bar{u}_c'''} - \bar{u}_{2c}'' - \frac{\bar{u}_c''''}{2}\left(\frac{\bar{u}_{1c}''}{\bar{u}_c'''}\right)^2\right\} - \frac{\bar{u}_{1c}'\bar{u}_1''}{\bar{u}_c'''} + \bar{u}_{2c}, \qquad (4.7a,b,c,d)$$
$$\tilde{\alpha}_1\bar{c} + \bar{\alpha}\tilde{c}_1 = 0,$$
$$\tilde{\alpha}_1\tilde{c}_1 + \tilde{\alpha}_2\bar{c} + \bar{\alpha}\tilde{c}_2 = 0.$$

The conditions (4.6), (4.7) impose a constraint on the initial profile $u = \bar{u}(y)$ if the Rayleigh wave is to stay neutral up to order X^2. Shortly we shall see how a small amplitude vortex emanating from $y = \bar{y}$ changes this conclusion. An examination of the passive boundary layer where $y = O(X^{1/3})$ shows that this layer does not influence (4.6), (4.7).

Now let us find the effect of a finite amplitude vortex on the above discussion. We recall from the work of Hall and Lakin (1988) that, if $G\bar{u}\bar{u}_y = 1$ at $x = 1$, then a finite amplitude vortex will initially grow within the region $-X^{1/2}\bar{y}_1 + \bar{y} < y < X^{1/2}\bar{y}_1 + \bar{y}$ for small values of X. A result of some consequence for the present investigation is that the mean flow modification driven by the vortex is $O(X^{3/2})$ and confined to the same region as the vortex. We shall choose B, the Rayleigh wave vortex amplitude to be of that size which also leads to a mean flow modification of size $X^{3/2}$. The corresponding effect of this mean flow correction on α and c can then be again deduced from (2.29) and (2.30). In fact the size of this effect can be found directly from (2.29) by noting that $\tilde{u}$ is now a function confined to a layer of depth $O(X^{1/2})$ well away from the critical layer. This means that the only contribution to (2.29) from $\tilde{u}$ comes from the terms in the integrand proportional to $\bar{P}'$. Since $\tilde{u} \sim O(X^{3/2})$ we should anticipate a contribution of this size to (2.29), however we shall see below that some cancellation in the integral occurs so that only a contribution of size $O(X^{5/2})$ is generated. Moreover since $\tilde{u}$ is exponentially small away from the critical layer, the order $X^{5/2}$ correction terms in the expansions of α and c must be identically zero. In this case the satisfaction of (2.29) is achieved by the adjustment of $B(X)$.

We define a variable θ by

$$\theta = \frac{(y - \bar{y})}{X^{1/2}}, \tag{4.13}$$

where

$$G\bar{u}\ \bar{u}y = 1,\ (\bar{u}\ \bar{u}y)y = 0, \quad y = \bar{y}. \tag{4.14}$$

We shall seek a solution of (2.16) - (2.20) by expanding

$$B = \bar{B}X^{1/2} + \cdots$$

and the velocity field in the form

$$\begin{aligned} u &= u_{00} + X^{1/2}\theta u_{10} + X\left\{\theta^2 u_{20} + u_{01}\right\} + X^{3/2}\theta u_{11} + X^{3/2}u_M(\theta) + \cdots, \\ v &= v_{00} + X^{1/2}\theta v_{10} + X\left\{\theta^2 v_{20} + v_{01}\right\} + X^{3/2}\theta v_{11} + X v_M(\theta) + \cdots, . \end{aligned} \tag{4.15a, b}$$

Here the constants u_{00}, etc. are defined by

$$\begin{aligned} &u_{00} = \bar{u}(\bar{y}), \quad u_{10} = \bar{u}'(\bar{y}) = \tfrac{1}{Gu_{00}}, \quad u_{20} = 2\bar{u}''(\bar{y}), \\ &u_{01} = \bar{u}_1(\bar{y}), \quad u_{11} = \bar{u}_1'(\bar{y}), \quad v_{00} = \bar{v}(\bar{y}), \quad v_{10} = \bar{v}'(\bar{y}), \\ &v_{20} = 2\bar{v}''(\bar{y}), \quad v_{01}(\bar{y}) = \bar{v}_1(\bar{y}), \quad v_{11} = \bar{v}_1'(\bar{y}). \end{aligned} \tag{4.16a - i}$$

Furthermore we assume that the wall curvature X expands locally as

$$\chi = \chi_0 + X\chi_1 + X^2\chi_2 + \cdots. \tag{4.17}$$

The Region III in this new notation is defined by

$$-\bar{\theta}X^{1/2} < \theta < \bar{\theta}X^{1/2}$$

whilst I, III are determined by $\theta > \bar{\theta}X^{1/2}$, $\theta < -\bar{\theta}X^{1/2}$ respectively. Here $\bar{\theta}$ is a constant to be determined and, because of symmetry, it will be sufficient for us to consider only $\theta > 0$.

In Region II we find, by equating successive powers of $X^{1/2}$ in (2.17a), that

$$G\chi_0 u_{00}u_{10} = 1,$$

$$u_{10}^2 + 2u_{00}u_{20} = 0,$$

$$\frac{-\chi_1}{G\chi_0^2 u_{00}} = \frac{\partial u_M}{\partial \theta} + u_{11} + \frac{3\theta^2 u_{10} u_{20}}{u_{00}} + \frac{u_{01} u_{10}}{u_{00}} - \bar{B}^2 \gamma,$$

where

$$\gamma = \frac{u_{10}^2}{[u_{00} - \bar{c}]} \left\{ \bar{P'}^2 (\bar{u} - \bar{c})^{-2} \right\}', \tag{4.18a, b, c, d}$$

evaluated at $y = \bar{y}$. Equations (4.18a,b) are automatically satisfied because of (4.14) and (4.18c) can be integrated once to give

$$u_M = \{\lambda_0 + \gamma \bar{B}^2\}\theta - \lambda_1 \theta^3, \tag{4.19}$$

with

$$\lambda_0 = -u_{11} - \frac{\chi_1 u_{10}}{\chi_0} - \frac{u_{01} u_{10}}{u_{00}}, \lambda_1 = \frac{u_{10} u_{20}}{u_{00}}. \tag{4.20}$$

We have assumed above that u_M is an odd function of θ as was found to be the case by Hall and Lakin (1988). Following Hall and Lakin we find that in Region I

$$u_M = \frac{(\lambda_0 \bar{\theta} + \gamma \bar{B}^2 \bar{\theta} - \lambda_1 \bar{\theta}^3) \exp\left\{-\frac{u_{00}}{8}[\theta^2 - \bar{\theta}^2]\right\} U\left(\frac{7}{2}, \sqrt{\frac{u_{00}}{2}}\theta\right)}{U\left(\frac{7}{2}, \sqrt{\frac{u_{00}}{2}}\bar{\theta}\right)}, \tag{4.21}$$

where we have already made u continuous at the junction of I,II and the continuity of u_y at the junction of I, II then yields

$$-\sqrt{\frac{u_{00}}{2}} \left\{ \frac{U\left(\frac{5}{2}, \sqrt{\frac{u_{00}}{2}}\bar{\theta}\right)}{U\left(\frac{7}{2}, \sqrt{\frac{u_{00}}{2}}\bar{\theta}\right)} \right\} = \frac{\lambda_0 + \gamma \bar{B}^2 - 3\lambda_1 \bar{\theta}^2}{\bar{\theta}[\lambda_0 + \gamma \bar{B}^2 - \lambda_1 \bar{\theta}^2]}. \tag{4.22}$$

In the absence of the Rayleigh wave forcing we can set $\bar{B} = 0$, equation (4.22) then reduces to the nonlinear eigenvalue problem for $\bar{\theta}$ found by Hall and Lakin (1988). Here equation (4.22) is not sufficient to determine $\bar{B}^2$ and $\bar{\Theta}$, the required extra condition is obtained by insisting that the $O(X^{3/2})$ mean flow correction in I, II, III does not make the Rayleigh wave wavenumber or wavespeed vary. The effect of the mean flow correction can be seen from equation (2.29). We expect that the term in (2.29) proportional to $\tilde{u}'$ will provide the dominant contribution to the integral, this would suggests that the dominant contribution to the integral would be of order $X^{\frac{3}{2}}$. However this contribution vanishes because u_M itself vanishes at infinity. Similarly the order X^2 contribution vanishes because u_M is an odd function of θ. Now we must set $\tilde{\alpha} = \tilde{c} = A = B = 0$ because the mean flow correction is exponentially small at the critical layer. Thus (2.29) leads to the equation

$$\int_{\infty}^{\infty} \theta u_M(\theta) d\theta = 0. \tag{4.23}$$

We note also that the higher order mean flow correction terms also vanish at infinity and are odd in θ and therefore do not contribute to (4.23). In the derivation of (4.23) we have used the result

$$\int_{-\infty}^{\infty} \theta^2 u_M'(\theta) d\theta = -2 \int_{-\infty}^{\infty} \theta u_M d\theta$$

Equation (4.23) yields

$$-\frac{6}{5\bar{\theta}^2 u_{00}}\left\{1+4\sqrt{\frac{u_{00}}{2}}\bar{\theta}\frac{U\left(\frac{9}{2},\sqrt{\frac{u_{00}}{2}}\bar{\theta}\right)}{U\left(\frac{7}{2},\sqrt{\frac{u_{00}}{2}}\bar{\theta}\right)}\right\}=\frac{\gamma\bar{B}^2+\lambda_0-\frac{3}{5}\lambda_1\bar{\theta}^2}{\gamma\bar{B}^2+\lambda_0-\lambda_1\bar{\theta}^2}. \tag{4.24}$$

Thus $\bar{\theta}$ and $\bar{B}$ are determined by the coupled system (4.22), (4.24), the system is simplified by writing

$$\Theta=\sqrt{\frac{u_{00}}{2}}\theta, \quad \lambda_1=\tilde{\lambda}_1\frac{u_{00}}{2}, \tag{4.25a, b}$$

in which case we obtain the coupled system

$$\frac{\gamma\bar{B}^2+\lambda_0-3\tilde{\lambda}_1\Theta^2}{\gamma\bar{B}^2+\lambda_0-\tilde{\lambda}_1\Theta^2}=-\frac{\Theta U\left(\frac{5}{2},\Theta\right)}{U\left(\frac{7}{2},\Theta\right)},$$

$$\frac{\gamma\bar{B}^2+\lambda_0-\frac{3}{5}\tilde{\lambda}_1\Theta^2}{\gamma\bar{B}^2+\lambda_0-\tilde{\lambda}_1\Theta^2}=-\frac{3}{5\Theta^2}\left\{1+4\frac{\Theta\, U\left(\frac{9}{2},\Theta\right)}{U\left(\frac{7}{2},\Theta\right)}\right\}, \tag{4.26a, b}$$

and if we multiply (4.26b) by 5 and add to (4.26a) we obtain

$$\frac{-3(1+\Theta^2)}{\Theta(3+\Theta^2)}=\frac{U\left(\frac{5}{2},\Theta\right)}{U\left(\frac{7}{2},\Theta\right)}. \tag{4.27}$$

The right hand side of the above equation is always positive so there are no roots of the equation for Θ positive. Since we have implicitly assumed above that θ is positive we conclude that there are no acceptable finite amplitude solutions of the interaction problem. Thus we have found that small wavelength vortices and Rayleigh waves cannot spontaneously be generated in a centrifugally and inviscidly unstable boundary layer. We conclude that the only possibility is that described in the previous section where we showed that small amplitude Rayleigh waves can be generated from an $O(1)$ vortex field.

5. Further discussion and conclusion

In Section 3 we saw that the initial stages of vortex-wave interactions in highly curved boundary layers can be expressed in terms of an asymptotic expansion in powers of the square root of the distance from the point of neutral stability. Our numerical investigations of a class of initial profiles showed that in some cases the interaction cannot occur because the predicted amplitude of the Rayleigh wave is imaginary. In nonlinear hydrodynamic instability theory this would usually suggest that a subcritical finite amplitude instability occurs. Here this is not the case because the blending layer structure fails when $x < 1$ because the parabolic cylinder functions now grow exponentially away from the transition layers.

Thus the solutions obtained in Section 3 with $\bar{B}^2$ negative cannot be used to begin a marching procedure to solve the full interaction equations. Nevertheless we believe that these solutions are still of some interest and are indeed of particular physical importance. This is because these solutions will play a crucial role when the streamwise lengthscale $x-1$

becomes sufficiently small for nonparallel effects to come into play; see Hall and Smith (1984) for a related discussion in connection with Tollmien-Schlichting waves in growing boundary layers. We do not address the nonparallel problem here but we note that nonparallel effects come into play when $x-1$ becomes so small that streamwise derivatives of the Rayleigh wave amplitude balance with changes in the amplitude induced by the variation of the mean state. In this case the interaction is governed by an integro-differential equation whose large x asymptotic form reduces to the small $x-1$ form found in Section 3. We believe that the negative $\bar{B}^2$ solutions found in Section 3 lead to solutions of the nonparallel evolution equations having a finite distance singularity. This would explain why they do not connect with small $x-1$ solutions, whilst we believe that the solutions obtained in Section 3 with positive $\bar{B}^2$ can be connected with solutions of the evolution equations.

In Section 4 we investigated the special case which occurs when the interaction begins with the vortex and Rayleigh wave both having small amplitudes. Here we found that the nonlinear eigenrelation (4.27) does not have any real solution so that the interaction cannot take place. We conclude that the spontaneous generation of Rayleigh waves and vortices cannot occur so that the only possibility is that discussed by Hall and Lakin (1988) with just a small amplitude vortex emerging from the neutral position. However there is again a nonparallel evolution problem associated with the analysis of Section 4 if X becomes sufficiently small. We expect that on the appropriately reduced streamwise lengthscale the structure outlined in Section 4 can be modified to show that the spontaneous generation of wave and vortices leads to a finite distance singularity.

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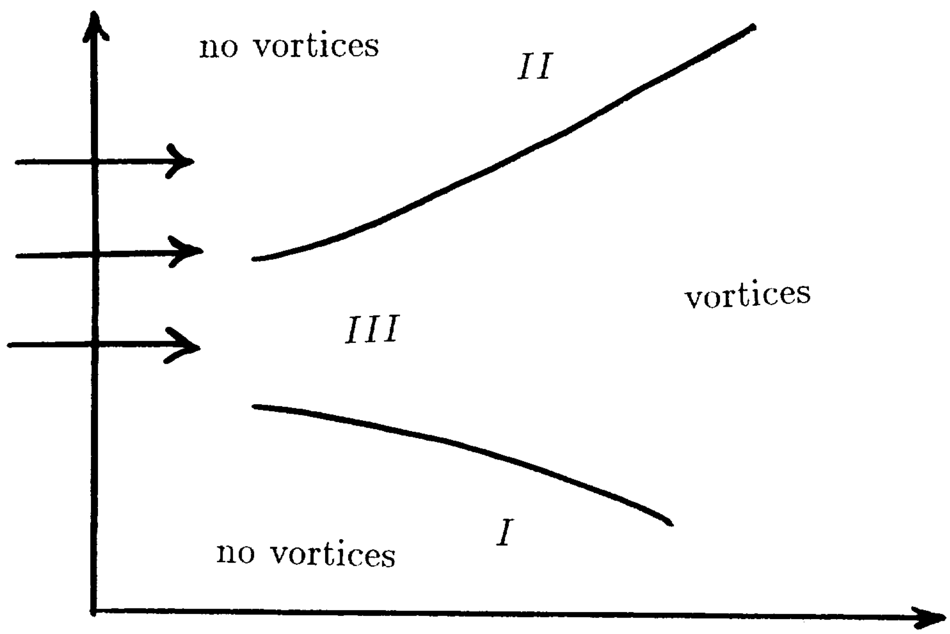

Figure 1.1. The different regimes in the small wavelength limit.

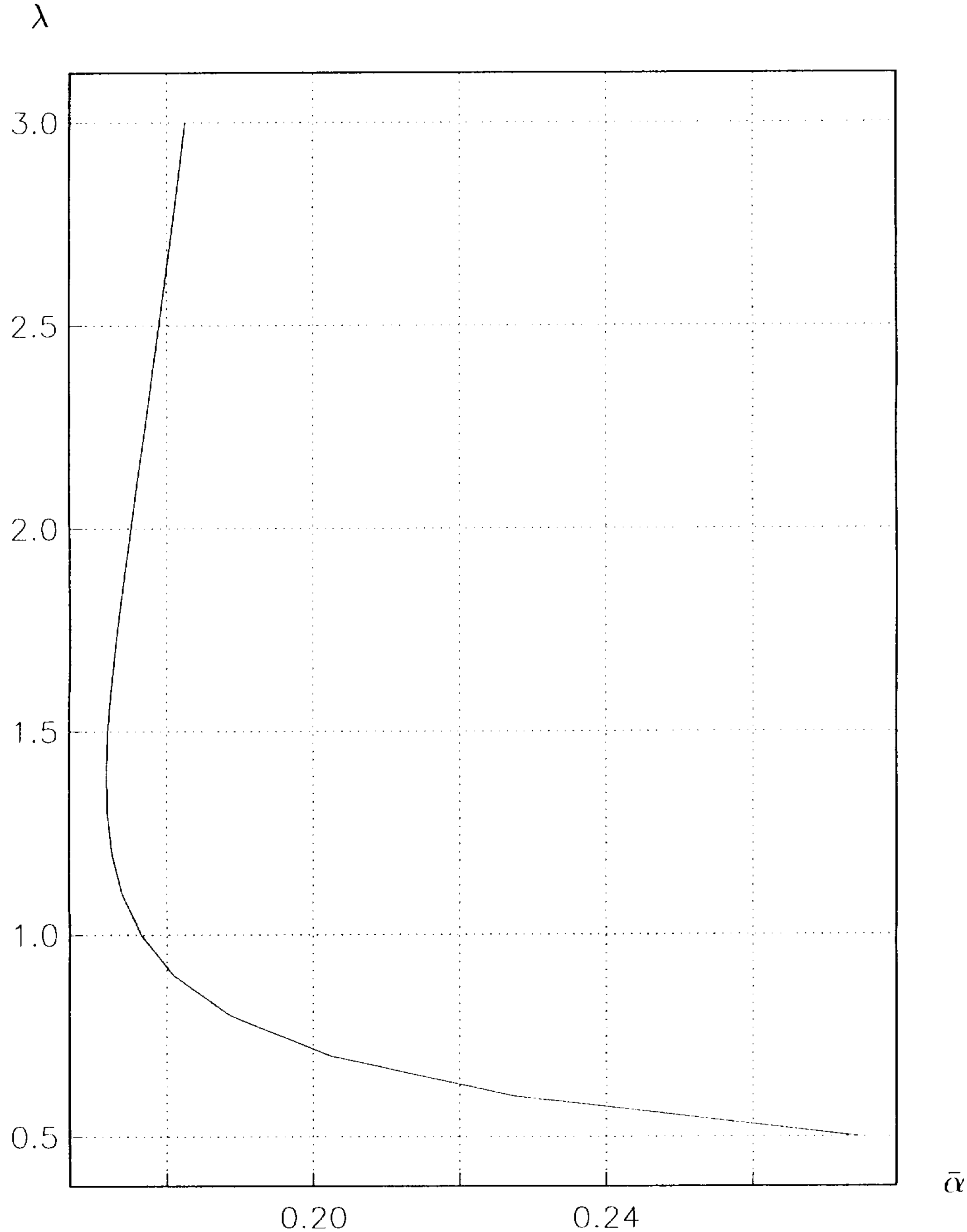

Figure 3.1. The dependence of $\bar{\alpha}$ on the quantity λ.

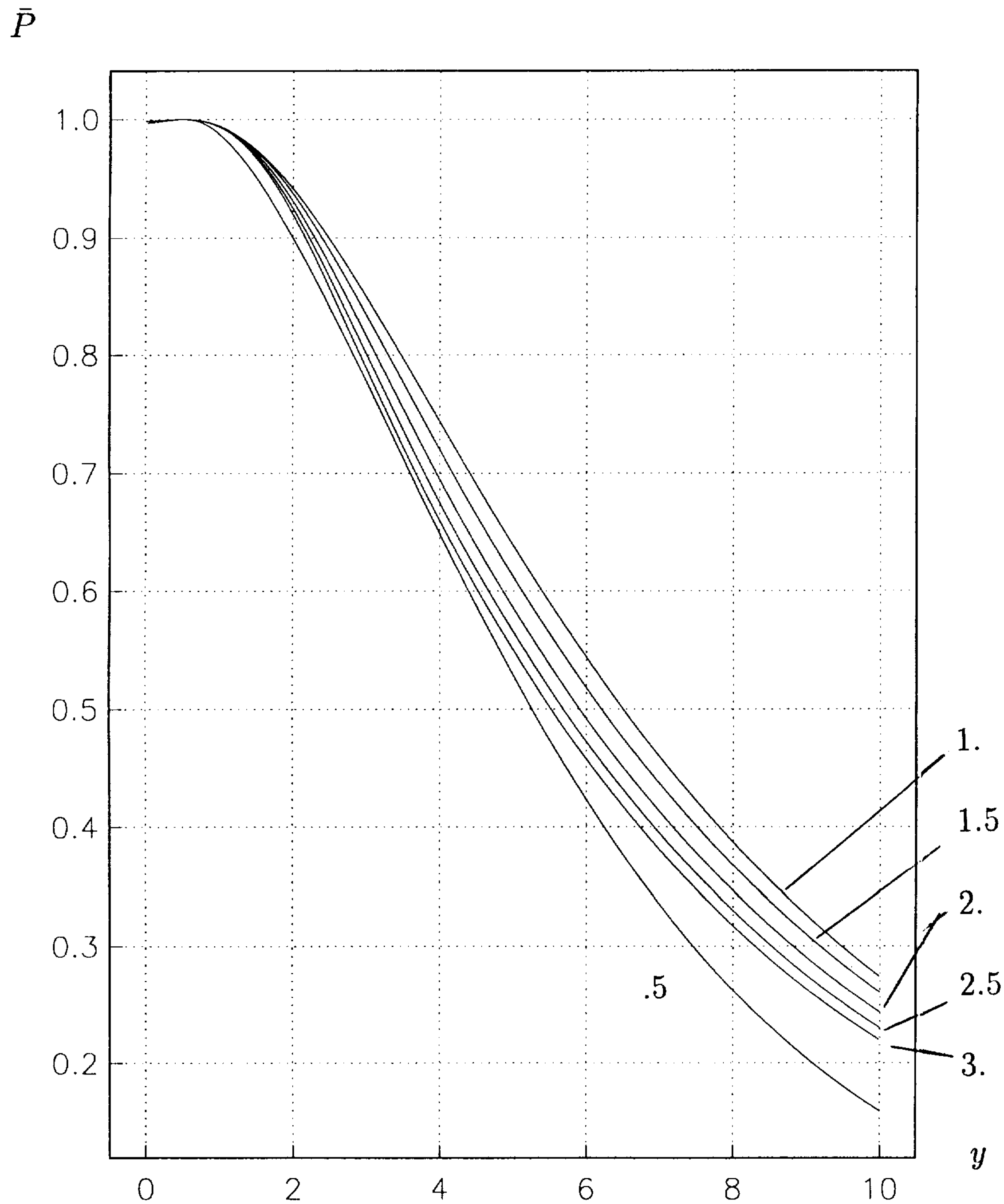

Figure 3.2. The eigenfunction P for different values of the quantity λ.

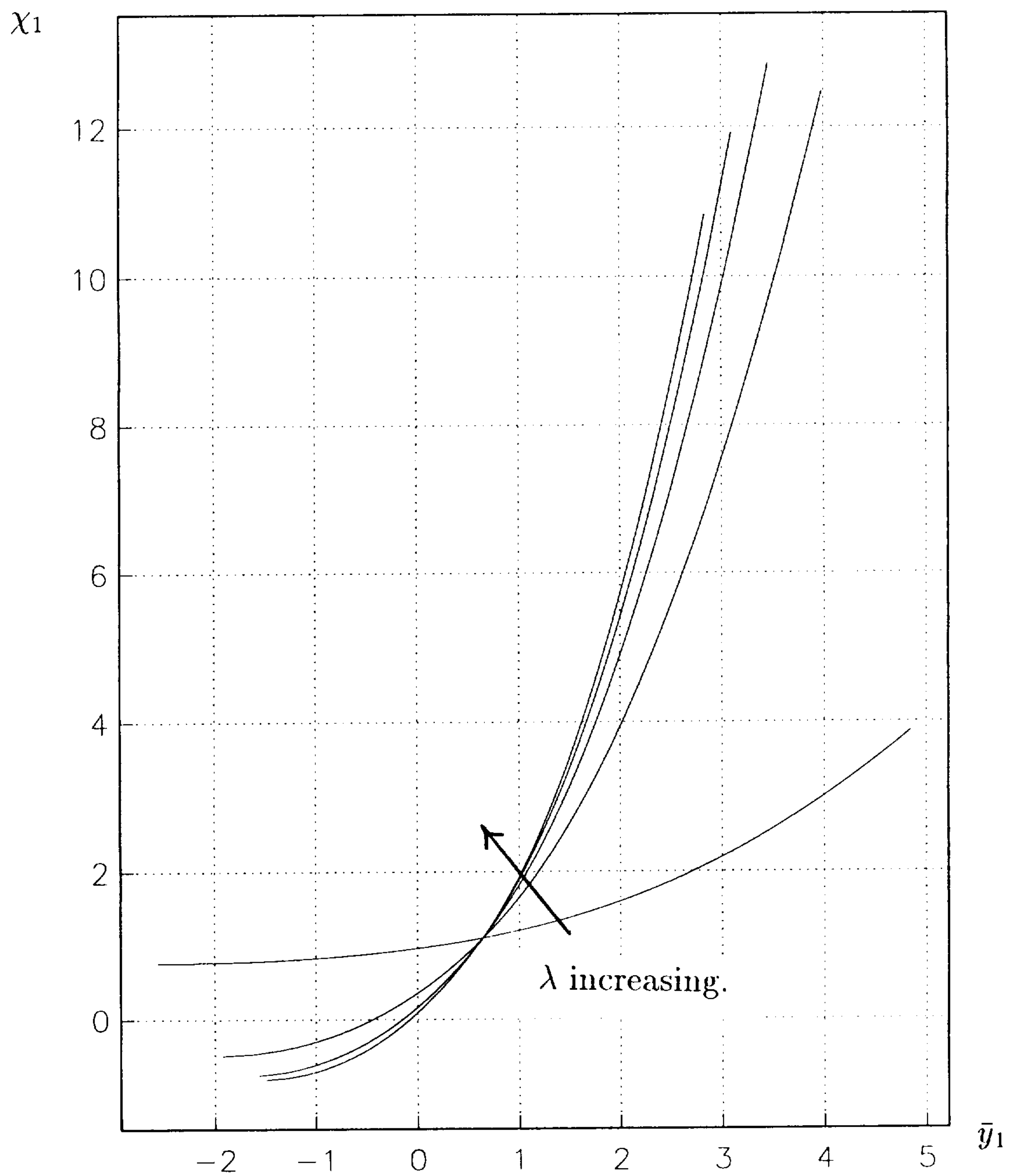

Figure 3.3a The quantity $\bar{y}_{11}$ as a function of χ_1 for $\lambda = 1, 1.5, 2, 2.5, 3$.

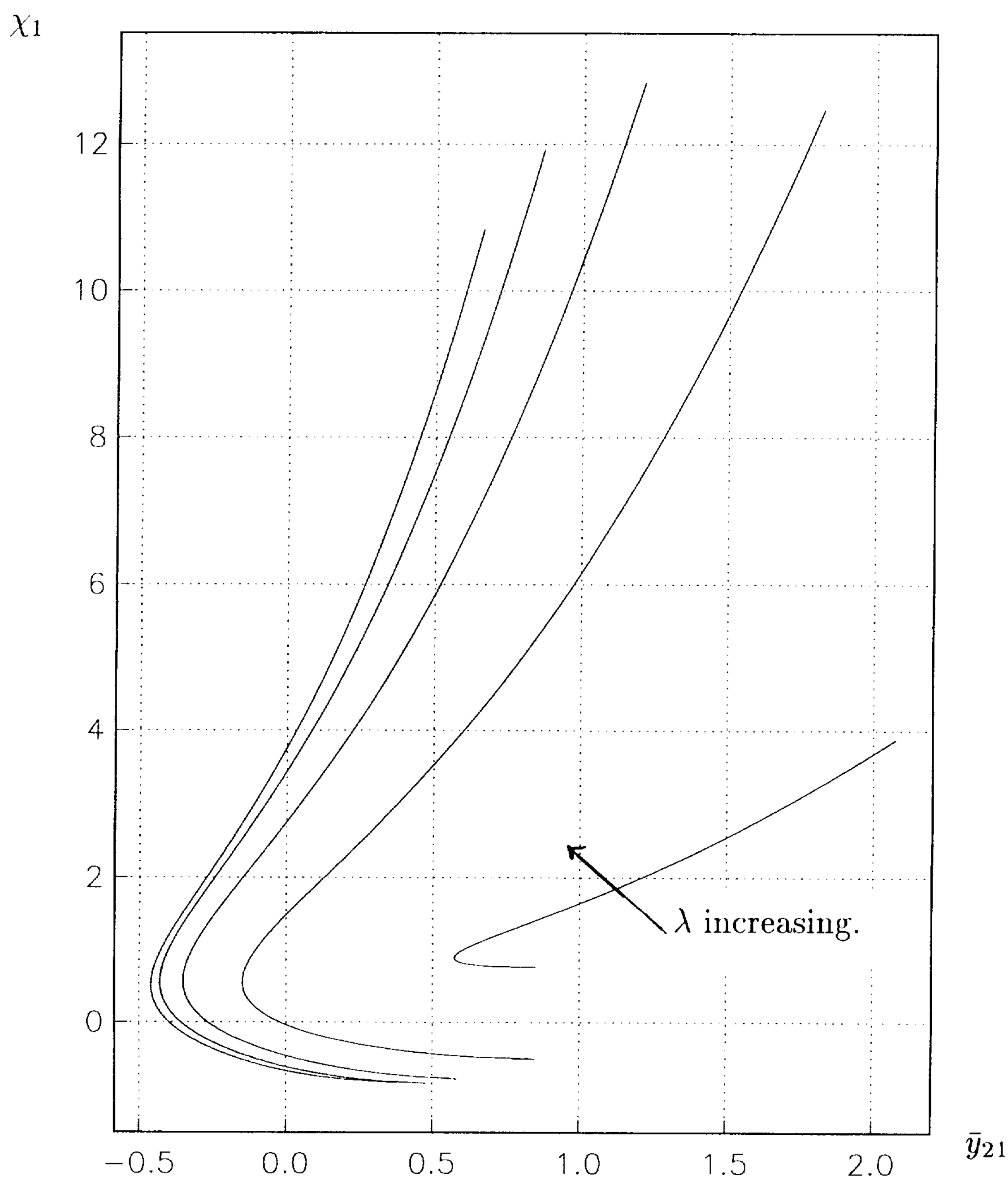

Figure 3.3b The quantity $\bar{y}_{21}$ as a function of χ_1 for $\lambda = 1, 1.5, 2, 2.5, 3$.

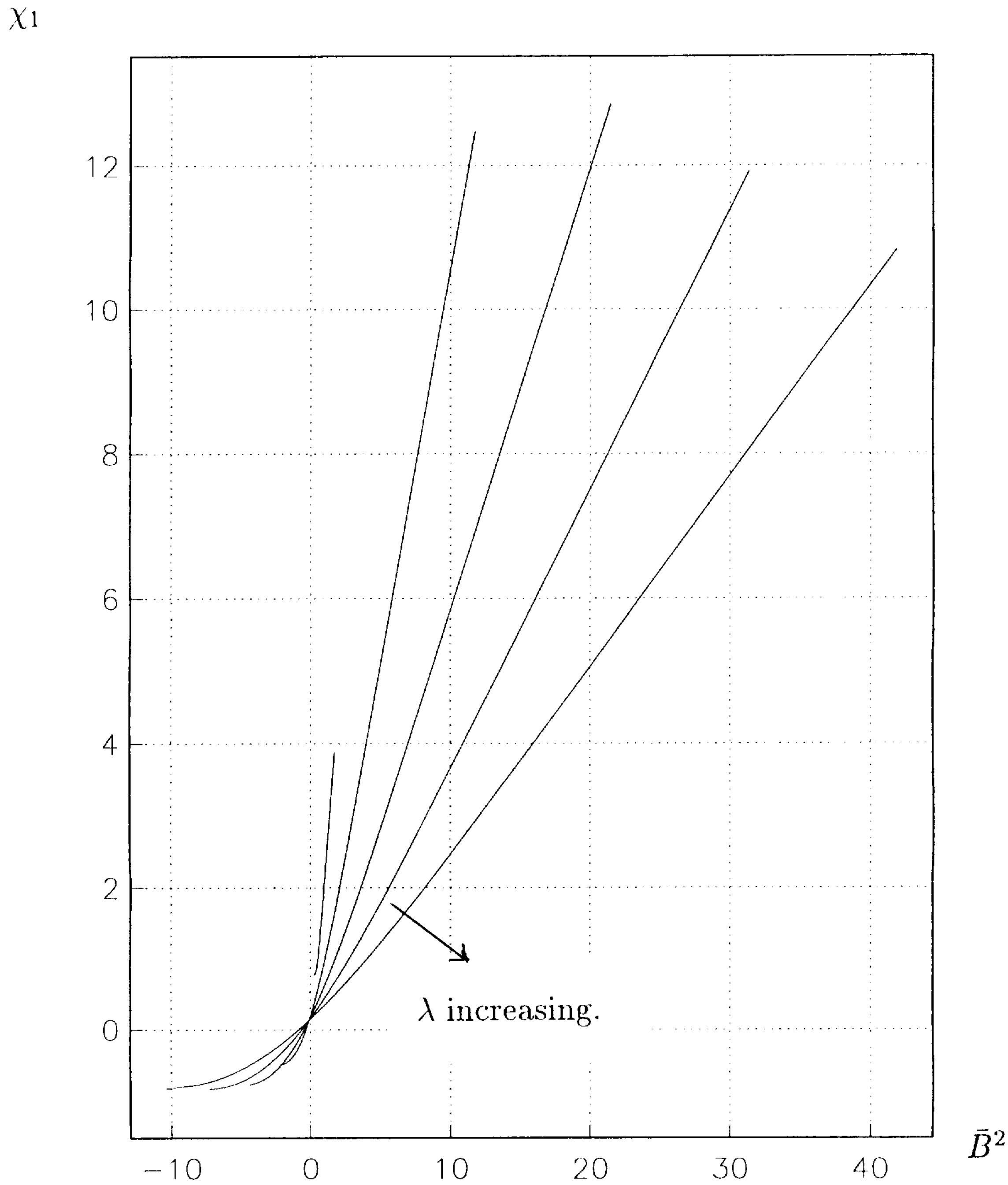

Figure 3.3c The quantity $\bar{B}^2$ as a function of χ_1 for $\lambda = 1, 1.5, 2, 2.5, 3$.

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13. ABSTRACT (*Maximum 200 words*)
The nonlinear interaction equations describing vortex-Rayleigh wave interactions in highly curved boundary layers are derived. These equations describe a strongly nonlinear interaction between an inviscid wave system and a streamwise vortex. The coupling between the two structures is quite different than that found by Hall and Smith (1991) in the absence of wall curvature. Here the vortex is forced over a finite region of the flow rather than in the critical layer associated with the wave system. When the interaction takes place the wave system remains locally neutral as it moves downstream and it's self interaction drives a vortex field of the same magnitude as that driven by the wall curvature. This modification of the mean state then alters the wave properties and forces the wave amplitude to adjust itself in order that the wave frequency is constant. Solutions of the interaction equations are found for the initial stages of the interaction in the case when the wave amplitude is initially small. Our analysis suggests that finite amplitude disturbances can only exist when the vortex field is finite at the initial position where the interaction is stimulated.

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